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ON LARGE DEVIATION PROBABILITIES FOR PROPERLY NORMALIZED WEIGHTED SUMS AND RELATED LAW OF ITERATED LOGARITHM

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ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with distribution function F. When F belongs to the domain of attraction of a stable law with index α, 0 <α < 2 and α≠1, an asymptotic behaviour of the large deviation probabilities with respect to properly normalized weighted sums have been studied and in support of this we obtained Chover's form of law of iterated logarithm.

Keywords: Large deviation probability, weighted sum, law of iterated logarithm

1.INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d) random variables (r.v.s) with distribution function (d.f) F, which belongs to the domain of attraction of a stable law with index α, 0<α<2 and α≠1. We denote this as F∈DA(α),0<α<2 and α≠1. Set $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$ and $T_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) X_k,$ where f is a non-decreasing and continuous on [0,1] and for any x∈[0,1], f(x)=1 gives us partial sums. Let $B_n = \inf\left\{x > 0 : 1 - F(x) + F(-x) \geq \frac{1}{n}\right\}$. Since F ∈ DA(α), 0<α<2 and α≠1, then we can have $B_n = n^{\frac{1}{\alpha}}\, l(n)$, where l is a function slowly varying (s.v) at ∞. When F ∈ DA(α),0<α<2 and α≠1, Beuerman [1975] proved that $\lim_{n \to \infty} P\left(\frac{T}{B_n}\right) = G(x)$,where G is a limiting stable law with index α, 0<α<2 and α≠1.Probability of large values plays an important role in studying the non-trivial limit behavior of stable like r.v.s. As far as properly normalized partial sums of stable like r.v.s, we can use the asymptotic results of Heyde [1968] for obtaining law of iterated logarithms (LIL) or rate of convergence problems [See Vasudeva [1978] and Gooty Divanji [2004]]. It is well known that the probabilities of the type $P(|S_n| > x_n)$ or either of the one sided components are called large deviation probabilities, where $\{x_n, n \geq 1\}$ is a monotone sequence of positive numbers with $x_n \to \infty$ as n→∞, such that $\frac{S_n}{x_n} \xrightarrow{P} 0$ as n→∞. In fact, under different conditions on the sequence of r.v.s, Heyde [[1967a],[1967b] and [1968]] studied large deviation problems for partial sums. In brief, when the underlying r.v.s are in the domain of attraction of a stable law, with index α, α≠1, Heyde [1968] obtained the precise asymptotic behaviour of large deviation probabilities.

For unit variance, Allan Gut [1986] studied the classical LIL for geometrically fast increasing subsequences of (S_n). In fact, he established that

$$\limsup_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} = \begin{cases} \sqrt{2} \text{ a.s., if } \limsup\limits_{k \to \infty} \frac{n_{k+1}}{n_k} < \infty \\ \varepsilon^* \text{ a.s., if } \liminf\limits_{k \to \infty} \frac{n_{k+1}}{n_k} > 1 \end{cases},$$

where $\varepsilon^* = \inf\left\{\varepsilon > 0 : \sum_{k=1}^{\infty} (\log n_k)^{\frac{-\varepsilon^2}{2}} < \infty\right\}$. Torrang [1987] extended to random subsequences. Observe that, when $n_k = 2^{2^k}$, then ε*=0 and we have $\limsup_{k \to \infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} = 0$ a.s.i.e, for such cases the norming sequence $\sqrt{n_k \log\log n_k}$ will not be precise

enough to give almost sure bound for (S_{n_k}). In general whenever $\frac{n_{k+1}}{n_k} \to \infty$, as $k \to \infty$, Schwabe and Gut [1996] have pointed out that $\sqrt{n_k \log\log n_k}$ is not a proper normalizing sequence and it has to be replaced by $\sqrt{n_k \log k}$. Note that $\frac{n_{k+1}}{n_k} \to \infty$, as $k \to \infty$, comes under the class of at least geometrically fast increasing subsequenes.

When n_k=n, Chover [1966] observed that in the case of stable r.v.s, LIL involving lim sup cannot be obtained under linear normalization and that it is possible under power normalization only. In fact, when X_n's are i.i.d. symmetric stable r.v.s, Chover [1966] established the LIL for (S_n), by normalizing in the power i.e., $\limsup_{n\to\infty} \left| \frac{S_n}{n^{\frac{1}{\alpha}}} \right|^{\frac{1}{\log\log n}} = e^{\frac{1}{\alpha}}$ a.s. Peng and Qi [2003] obtained Chover's type LIL for weighted sums of i.i.d r.v.s which are in the domain of attraction of a stable law with index α, 0<α<2, where the weights belong to Bounded Variation on [0,1], we denote the same as BV[0,1]. Many authors studied the non-trivial limit behavior for weighted sums. (See Vasudeva [1978] and Peng and Qi [2003]).

However, the observations made by Heyde [1967b] on the large deviation probabilities implicitly motivated us to study the large deviation probabilities for weighted sums and obtain a non-trivial limit behavior of properly normalized weighted sums for subsequences.

In the next section we present some lemmas and main results are presented in section 3. In the last section, we discuss the existence of Chover's LIL for weighted sums for subsequences. In the process, i.o, a.s and s.v mean 'infinitely often', 'almost surely' and 'slowly varying' respectively. C, ε, k and n with or without a super script or subscript denote positive constants with k and n confined to be integers.

2. Lemmas

Lemma 1 [Drasin and Seneta [1986]]

Let L be any s.v. function and let (x_n) and (y_n) be sequences of real constants tending to ∞ as n→∞. Then for any $\delta > 0$, $\lim_{n\to\infty} y_n^{\delta} \frac{L(x_n y_n)}{L(x_n)} = \infty$ and $\lim_{n\to\infty} y_n^{-\delta} \frac{L(x_n y_n)}{L(x_n)} = 0$.

Lemma 2 [Vasudeva and Divanji [1991]]

Let F ϵ DA(α),0<α<2. Let (x_n) be a monotone sequence of real numbers tending to ∞, as n→∞. Then $\frac{S_n}{x_n B_n} \xrightarrow{P} 0$, as $n \to \infty$, where

$B_n = n^{\frac{1}{\alpha}} l(n)$ and l is s.v. at ∞.

Lemma 3

Let F ϵ DA(α),0 <α< 2 and α≠1. Let (x_n) be a monotone sequence of real numbers tending to ∞, as n→∞. Then $\frac{T_n}{x_n B_n} \xrightarrow{P} 0$, as $n \to \infty$, where $B_n = n^{\frac{1}{\alpha}} l(n)$ and l is s.v. at ∞ and $T_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) X_k$, f is a non-decreasing and continuous function on [0,1].

Proof

Since f is a non-decreasing and continuous function on [0,1] and by Abel's partial summation formula, we have

$$\begin{aligned}\sum_{k=1}^{n} f\left(\tfrac{k}{n}\right) X_k &= \sum_{k=1}^{n} \left(f\left(\tfrac{k}{n}\right) - f\left(\tfrac{k-1}{n}\right) \right) S_k + f(0) S_0 \\ &\le \max_{1\le k\le n} S_k \left(f(1) - f(0) \right) \\ &\le S_n \left(f(1) - f(0) \right) \qquad (1)\end{aligned}$$

We have

$T_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) X_k \le S_n \left(f(1) - f(0) \right) = S_n$. Dividing on both sides by $x_n B_n$, we have $\frac{T_n}{x_n B_n} \le \frac{S_n}{x_n B_n}$. By Lemma 2, we have $\frac{S_n}{x_n B_n} \xrightarrow{P} 0$ and this leads to $\frac{T_n}{x_n B_n} \xrightarrow{P} 0$, as $n \to \infty$.

3. Main results

Theorem 1

Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d r.v.s with a d.f F and assume that FϵDA(α),0<α<2 and α≠1. Let (x_n) be a

monotone sequence of positive numbers with $x_n \to\infty$, as $n\to\infty$. Then $\lim\limits_{n\to\infty}\frac{P(T_n\ge x_nB_n)}{nP(X\ge x_nB_n)} = 1$, where $T_n=\sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right)X_k$, and f is non-decreasing and continuous function on[0,1] with f(1)=1 and f(0)=0, $B_n = n^{\frac{1}{\alpha}}l(n)$ and l is s.v. at ∞.

Proof

To prove the assertion, first we show that $\liminf\limits_{n\to\infty}\frac{P(T_n\ge x_nB_n)}{nP(X\ge x_nB_n)}\ge 1$ and later we establish that $\limsup\limits_{n\to\infty}\frac{P(T_n\ge x_nB_n)}{nP(X\ge x_nB_n)}\le 1$ Observe that by (1), we have

$$P\left(T_n\ge x_nB_n\right)\ge P(S_n\left(f(1)\text{-}f(0)\right)\ge x_nB_n)$$
$$\ge P(S_n\ge\left(f(1)\text{-}f(0)\right)^{-1}x_nB_n)$$

which implies

$\frac{P(T_n\ge x_nB_n)}{nP(X\ge x_nB_n)}\ge\frac{P\left(S_n\ge(f(1)\text{-}f(0))^{-1}x_nB_n\right)}{nP(X\ge x_nB_n)}$. It is well known that when F ϵ DA(α),0<α<2 and α≠1, [See Mijnheer [1975] of Theorem 2.2 on page 16], we have

$$\lim_{x\to\infty}\frac{(1\text{-}F(x)+F(\text{-}x))}{x^{-\alpha}L(x)} = 1 \qquad (2)$$

Hence using (2), we get,

$$\frac{P(T_n\ge x_nB_n)}{nP(X\ge x_nB_n)}\ge\frac{P\left(S_n\ge(f(1)\text{-}f(0))^{-1}x_nB_n\right)}{nP(X\ge x_nB_n)}$$
$$\ge\frac{n\left((f(1)\text{-}f(0))^{-1}x_nB_n\right)^{-\alpha}}{n\left(x_nB_n\right)^{-\alpha}}\frac{L\left((f(1)\text{-}f(0))^{-1}x_nB_n\right)}{L\left(x_nB_n\right)}$$
$$\ge\left(f(1)\text{-}f(0)\right)^{-\alpha},$$

for large n (by Lemma 1). Since f is non-decreasing and continuous on [0,1] and for large n, we can get that $\liminf\limits_{n\to\infty}\frac{P(T_n\ge x_nB_n)}{nP(X\ge x_nB_n)}\ge 1.$

In order to complete the proof, we use truncation method. Define

$$Y_k = \begin{cases} X_k, & \text{if } f\left(\frac{k}{n}\right)X_k\le x_nB_n\\ 0, & \text{otherwise}\end{cases}.$$

Let $R_k = f\left(\frac{k}{n}\right)X_k - f\left(\frac{k}{n}\right)Y_k$,

$$T_{1,n}=\sum_{k=1}^{n} f\left(\frac{k}{n}\right)Y_k \text{ and } T_{2,n}=\sum_{k=1}^{n}R_k.$$

Note that

$$P\left(T_n\ \ge\ x_nB_n\right)\ \le\ P\left(T_{1,n} > x_nB_n\right)$$
$$+\ P\left(T_{2,n}\ne 0\right).$$

This implies

$$\frac{P\left(T_n\ge x_nB_n\right)}{nP\left(X\ge x_nB_n\right)}\le\frac{P\left(T_{1,n}\ge x_nB_n\right)}{nP\left(X\ge x_nB_n\right)}+\frac{P\left(T_{2,n}\ne 0\right)}{nP\left(X\ge x_nB_n\right)} \qquad (3)$$

Observe that

$$P\left(T_{2,n}\ne 0\right)=P\left(\sum_{k=1}^{n}R_k\ne 0\right)$$
$$\le\sum_{k=1}^{n}P(R_k\ne 0)$$
$$\le\sum_{k=1}^{n}P\left(f\left(\frac{k}{n}\right)X_k\ge x_nB_n\right).$$

Since f is non-decreasing and continuous on [0,1] and for all k, $1\le k\le n$, we may get that

$$f\left(\frac{1}{n}\right)\le f\left(\frac{k}{n}\right)\le f(1) \text{ or } f\left(\frac{1}{n}\right)\le f(1). \qquad (4)$$

Using the fact (4), one can find some k such that for all $k\ge k_1$, we have

$$P\left(T_{2,n}\ne 0\right)\le\sum_{k=1}^{n}P\left(X_k\ge f^{-1}(1)x_nB_n\right)$$
$$\le nP\left(X\ge f^{-1}(1)x_nB_n\right),$$

since X_k's are i.i.dr.v.s. Using (2), we have

$$\frac{P\left(T_{2,n}\ne 0\right)}{nP\left(X\ge x_nB_n\right)}\le\frac{nP\left(X\ge f^{-1}(1)x_nB_n\right)}{nP\left(X\ge x_nB_n\right)}$$
$$\le f^{\alpha}(1)\frac{L\left(f^{-1}(1)x_nB_n\right)}{L\left(x_nB_n\right)}$$

Again using Karamata's representation of s.v. function at ∞, one gets that

$$\frac{L\left(f^{-1}(1)x_nB_n\right)}{L(x_nB_n)}$$

$$=\frac{a\left(f^{-1}(1)x_nB_n\right)}{a(x_nB_n)}\exp\left\{\int_0^{f^{-1}(1)x_nB_n}\frac{\varepsilon(y)}{y}dy-\int_0^{x_nB_n}\frac{\varepsilon(y)}{y}dy\right\}$$

$$=\frac{a\left(f^{-1}(1)x_nB_n\right)}{a(x_nB_n)}\exp\left\{\int_{x_nB_n}^{f^{-1}(1)x_nB_n}\frac{\varepsilon(y)}{y}dy\right\}$$

Since $\varepsilon(y)\to 0$ and $a(y)\to C<\infty$ as $y\to\infty$, there exists $C_2>0$ and $\delta_0<\alpha$, such that

$$\frac{a\left(f^{-1}(1)x_nB_n\right)}{a(x_nB_n)}\le C_2,\ \varepsilon(y)\le\delta_0,\ \text{for } y\ \ge x_nB_n.$$ This

yields

$$\frac{L\left(f^{-1}(1)x_nB_n\right)}{L(x_nB_n)}\le C_2\exp\left\{\delta_0\log\left(f^{-1}(1)\right)\right\}$$

$$\le\frac{C_2}{f^{\delta_0}(1)}.$$

Using the fact that f is non-decreasing and continuous on [0,1] and for some constant C_3, one gets that

$$\operatorname*{Lim}_{n\to\infty}\frac{P\left(T_{2,n}\ne 0\right)}{nP\left(X\ge x_nB_n\right)}\le\operatorname*{Lim}_{n\to\infty}\left(f(1)\right)^{\alpha-\delta_0}=C_3 \qquad (5)$$

Now consider the first term in the right of (3). By Tchebychev's inequality, we get,

$$\frac{P\left(T_n\ge x_nB_n\right)}{nP\left(X\ge x_nB_n\right)}\le\frac{E\left(T_{1,n}^2\right)}{nx_n^2B_n^2P(X\ge x_nB_n)}.$$ Since

$$E(T_{1,n}^2)=\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)EY_k^2+\sum_{\substack{k=1\\k\ne m}}^{n}\sum_{m=1}^{n}f\left(\frac{k}{n}\right)f\left(\frac{m}{n}\right)EY_kEY_m$$

We have

$$\frac{P\left(T_{1,n}\ge x_nB_n\right)}{nP\left(X\ge x_nB_n\right)}\le\frac{E\left(T_{1,n}^2\right)}{nx_n^2B_n^2P(X\ge x_nB_n)}$$

$$\le\frac{\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)EY_k^2}{nx_n^2B_n^2P(X\ge x_nB_n)}+\frac{\sum_{\substack{k=1\\k\ne m}}^{n}\sum_{m=1}^{n}f\left(\frac{k}{n}\right)f\left(\frac{m}{n}\right)EY_kEY_m}{nx_n^2B_n^2P(X\ge x_nB_n)} \qquad (6)$$

By Theorem 1 on page 544 of Feller [1986, Vol. II] and (2), one gets that

$$\frac{\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)EY_k^2}{nx_n^2B_n^2P(X\ge x_nB_n)}$$

$$\le\frac{x_n^{\alpha}B_n^{\alpha}\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)f^{\alpha-2}\left(\frac{k}{n}\right)x_n^{2-\alpha}B_n^{2-\alpha}L\left(f^{-1}\left(\frac{k}{n}\right)x_nB_n\right)}{nx_n^2B_n^2L(x_nB_n)}$$

$$\le\frac{1}{n}\sum_{k=1}^{n}f^{\alpha}\left(\frac{k}{n}\right)\frac{L\left(f^{-1}\left(\frac{k}{n}\right)x_nB_n\right)}{L(x_nB_n)}$$

Using Karamata's representation of s.v. function at ∞, one gets that

$$\frac{L\left(f^{-1}(1)x_nB_n\right)}{L\left(x_nB_n\right)}$$

$$=\frac{a\left(f^{-1}(1)x_nB_n\right)}{a\left(x_nB_n\right)}\exp\left\{\int_0^{f^{-1}(1)x_nB_n}\frac{\varepsilon(y)}{y}dy-\int_0^{x_nB_n}\frac{\varepsilon(y)}{y}dy\right\}$$

$$=\frac{a\left(f^{-1}(1)x_nB_n\right)}{a\left(x_nB_n\right)}\exp\left\{\int_{x_nB_n}^{f^{-1}(1)x_nB_n}\frac{\varepsilon(y)}{y}dy\right\}$$

Since $\varepsilon(y)\to 0$ and $a(y)\to C<\infty$ as $y\to\infty$, there exists $C_4>0$ and $\delta_0<\alpha$, such that

$$\frac{a\left(f^{-1}(1)x_nB_n\right)}{a\left(x_nB_n\right)}\to C_4,\ \varepsilon(y)\ge -\delta_0,\ \text{for } y\ge x_nB_n.$$ This gives

$$\frac{L\left(f^{-1}(1)x_nB_n\right)}{L\left(x_nB_n\right)}\ge C_4\exp\left\{-\delta_0\log\left(\frac{1}{f(1)}\right)\right\}\ge C_1\left(f^{\delta_0}(1)\right) \qquad (7)$$

From (7), one can find a constant C_5 (> 0) such that

$$\frac{\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)EY_k^2}{nx_n^2B_n^2P\left(X\ge x_nB_n\right)}\le\frac{C_5}{n}\sum_{k=1}^{n}f^{\alpha-\delta_0}\left(\frac{k}{n}\right)$$

By the assumption (4), we can find a constant C_6 (> C_5) such that

$$\frac{\sum_{k=1}^{n}f^2\left(\frac{k}{n}\right)EY_k^2}{nx_n^2B_n^2P\left(X\ge x_nB_n\right)}\le C_6 \qquad (8)$$

Observe that

$$\sum_{\substack{k=1 \\ k\neq m}}^{n}\sum_{m=1}^{n} f\left(\frac{k}{n}\right) f\left(\frac{m}{n}\right) EY_k EY_m \le \left\{\sum_{k=1}^{n} f\left(\frac{k}{n}\right)|EY_k|\right\}^2$$

Now for $0 < \alpha < 1$,

$$|EY_k| \le E|Y_k| = \le \int_{|x|\le f^{-1}\left(\frac{k}{n}\right)x_nB_n} |x|dP(X<x) \le \int_0^{f^{-1}\left(\frac{k}{n}\right)x_nB_n} P(X \ge x)dx$$

Let $A = \dfrac{\sum_{\substack{k=1 \\ k\neq m}}^{n}\sum_{m=1}^{n} f\left(\frac{k}{n}\right) f\left(\frac{m}{n}\right) EY_k EY_m}{nx_n^2B_n^2P\left(X \ge x_nB_n\right)}$,

$$B = \frac{\left\{\sum_{k=1}^{n} f\left(\frac{k}{n}\right)|EY_k|\right\}^2}{nx_n^2B_n^2P\left(X \ge x_nB_n\right)}$$

and $D = \dfrac{\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\int_0^{f^{-1}\left(\frac{k}{n}\right)x_nB_n} P(X\ge x)dx\right)^2}{nx_n^2B_n^2P\left(X \ge x_nB_n\right)}$

Note that $A \le B \le D$. Again using (2), we get,

$$D \le \frac{\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\int_0^{f^{-1}\left(\frac{k}{n}\right)x_nB_n} x^{-\alpha}L(x)dx\right)^2}{nx_n^{2-\alpha}B_n^{2-\alpha}L\left(x_nB_n\right)} \le \frac{\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\int_0^{f^{-1}\left(\frac{k}{n}\right)x_nB_n} x^{-\alpha}\frac{L(x)}{L(x_nB_n)}dx\right)^2}{nx_n^{2-\alpha}B_n^{2-\alpha}} L\left(x_nB_n\right).$$

Following similar steps of (5), we can find some constant $C_7$and $\delta_0>0$ such that $\dfrac{L(x)}{L(x_nB_n)} \le C_7\left(1+\delta_0\right)\left(\dfrac{x_nB_n}{x}\right)^{\delta_0}$. Substituting these facts we get,

$$D \le \frac{\left(C_7\left(1+\delta_0\right)\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\int_0^{f^{-1}\left(\frac{k}{n}\right)x_nB_n} x^{-\alpha-\delta_0}dx\, x_n^{\delta_0}B_n^{\delta_0}\right)^2}{nx_n^{2-\alpha}B_n^{2-\alpha}} L\left(x_nB_n\right)$$

Since $\int_0^{f^{-1}\left(\frac{k}{n}\right)x_nB_n} x^{-\alpha-\delta_0}dx = \frac{1}{1-\alpha-\delta_0} x_n^{1-\alpha-\delta_0}B_n^{1-\alpha-\delta_0}f^{\alpha+\delta_0-1}\left(\frac{k}{n}\right)$.

Hence there exists $C_8(> C_7)$ such that

$$D \le \frac{C_8\left(\sum_{k=1}^{n} f^{\alpha+\delta_0}\left(\frac{k}{n}\right)\right)^2 L(x_nB_n)}{nx_n^{\alpha}B_n^{\alpha}}.$$

Consider $\dfrac{L(x_nB_n)}{nx_n^{\alpha}B_n^{\alpha}} = \dfrac{L(B_n)}{nx_n^{\alpha}B_n^{\alpha}}\dfrac{L(x_nB_n)}{L(B_n)}$ and using Lemma 1, we get, $\dfrac{L(x_nB_n)}{nx_n^{\alpha}B_n^{\alpha}} \le \dfrac{L(B_n)}{nx_n^{\alpha}B_n^{\alpha}}x_n^{\delta} \le \dfrac{nL(B_n)}{B_n^{\alpha}}\dfrac{1}{n^2x_n^{\alpha-\delta}}$. Since F ϵ DA(α), 0 < α <2 and α ≠ 1, we know that, for some $C_9>0$, $\dfrac{nL(B_n)}{B_n^{\alpha}} \to C_9$ and choose δ< α such that there exists some constant C_{10} (> C_9) such that $\dfrac{L(x_nB_n)}{nx_n^{\alpha}B_n^{\alpha}} \to C_{10}$.

Therefore $D \le C_{10}\sum_{k=1}^{n}\left(f^{\alpha+\delta_0}\left(\frac{k}{n}\right)\right)^2$. Since f(x) is non-decreasing and continuous on [0,1], one can find C_{11} such that $\sum_{k=1}^{n} f^{\alpha+\delta_0}\left(\frac{k}{n}\right) \le nC_{11}$ and hence for some C_{12} (> C_{11}), we have

$$D \le C_{12} \Rightarrow B \le C_{12} \Rightarrow A \le C_{12}. \qquad (9)$$

On the other hand, if 1 < α < 2 and $EX_1=0$ then $|EY_k| = \left|\int_{f^{-1}\left(\frac{k}{n}\right)x_nB_n}^{\infty} xdF(x)\right|$. Majoring $|EY_k|$ by $\int_{f^{-1}\left(\frac{k}{n}\right)x_nB_n}^{\infty} P\left(X \le x\right)dx$ and following similar steps of the case 0 < α < 1, we can able to show that

$$\frac{\sum_{\substack{k=1 \\ k\neq m}}^{n}\sum_{m=1}^{n} f\left(\frac{k}{n}\right) f\left(\frac{m}{n}\right) EY_k EY_m}{nx_n^2B_n^2P\left(X \ge x_nB_n\right)} \to 0 \text{ as } n \to \infty$$

(10)

From (8), (9) and (10), we claim that, $\frac{P\left(T_{1,n} \geq x_n B_n\right)}{nP\left(X \geq x_n B_n\right)} \to 0, \text{ as } n \to \infty.$

(11)

Substituting (5) and (111) in (3), we get $\operatorname*{Lim\,sup}_{n\to\infty} \frac{P\left(T_n \geq x_n B_n\right)}{nP\left(X \geq x_n B_n\right)} \geq 1$, and the proof of the theorem is completed.

CHOVER'S FORM OF LIL FOR SUBSEQUENCE OF PROPERLY NORMALIZEDWEIGHTED SUMS

Theorem 2

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d positive r.v.s with a d.f F and assume that F ϵ DA(α),0<α<1. Let $T_{n_k} = \sum_{k=1}^{n} f\left(\frac{k}{n_k}\right) X_{n_k}$, where f is a positive, non-decreasing and continuous function on [0,1] . Let $\{n_k\}$ be an integer subsequence such that $\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$.

Then

$$\operatorname*{Lim\,sup}_{n\to\infty} \left(\frac{T_{n_k}}{n_k^{\frac{1}{\alpha}}}\right)^{\frac{1}{\log\log n_k}} = e^{\frac{\varepsilon^*}{\alpha}} \text{ a.s,}$$

$$\text{where } \varepsilon^* = \inf\left\{\varepsilon > 0 : \sum_{k=k_0}^{\infty} \left(\log n_k\right)^{-\varepsilon} < \infty\right\}.$$

In particular, if $\operatorname*{Lim}_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty$, as $k \to \infty$, but not limit is ∞ and $\varepsilon^*=0$, then

$$\operatorname*{Lim\,sup}_{n\to\infty} \left(\frac{T_{n_k}}{n_k^{\frac{1}{\alpha}}}\right)^{\frac{1}{\log k}} = e^{\frac{1}{\alpha}} \text{ a.s.}$$

Proof

To prove the assertion, it suffices to show for any ε ϵ (0,1) that

$$P\left(T_{n_k} \geq n_k^{\frac{1}{\alpha}} \left(\log n_k\right)^{\frac{\varepsilon^*+\varepsilon}{\alpha}} \text{ i.o}\right) = 0 \qquad (12)$$

and

$$P\left(T_{n_k} \geq n_k^{\frac{1}{\alpha}} \left(\log n_k\right)^{\frac{\varepsilon^*-\varepsilon}{\alpha}} \text{ i.o}\right) = 1 \qquad (13)$$

To prove (12),

let $M_k = \left\{T_{n_k} \geq n_k^{\frac{1}{\alpha}} \left(\log n_k\right)^{\frac{\varepsilon^*+\varepsilon}{\alpha}}\right\}$ and

$y_{n_k} = n_k^{\frac{1}{\alpha}} \left(\log n_k\right)^{\frac{\varepsilon^*+\varepsilon}{\alpha}}$.

By the Theorem 1, one can find a C_1 and a k_1 such that, for all $k \geq k_1$, $P(M_k) \leq C_1 n_k P(X \geq y_{n_k})$. It is well known that

$F \in DA(\alpha)$, $0 < \alpha < 1$, the equation (2)

becomes $\operatorname*{Lim}_{n\to\infty} \frac{1-F(x)}{x^{-\alpha}L(x)} = 1$ and using this fact, one can find a k_2 ($\geq k_1$) such that for all $k \geq k_2$,

$$P(M_k) \leq C_1 n_k y_{n_k}^{-\alpha} L(y_{n_k})$$

$$\leq \frac{C_1 n_k L(y_{n_k})}{n_k \left(\log n_k\right)^{(\varepsilon^*+\varepsilon)}} \frac{L\left(n_k^{\frac{1}{\alpha}}\right)}{L\left(n_k^{\frac{1}{\alpha}}\right)}.$$

Using Lemma 1, with $\delta = \frac{\varepsilon}{2}$, we choose ε sufficiently small and by the definition of ε^*, one can find k_3($\geq k_2$) such that for all k($\geq k_3$), $P(M_k) \leq C_2 (\log n_k)^{-\left(\varepsilon^* + \frac{\varepsilon}{2}\right)}$ for some $C_2 > 0$.

Consequently, $\sum_{k=k_3}^{\infty} P(M_k) < \infty$ and (13) follows from Borel-Cantelli lemma.

Using the relation $T_{n_k} = T_{n_k} - T_{n_{k-1}} + T_{n_{k-1}}, k \geq 1$ and define, for large k, $m_k = \min\left\{j : n_j \geq \beta^{(k-1)^{\delta}}\right\}$ (14)

where β>1 and δ>0. In order to establish (13), it is enough, if we show that for ε϶(0,ε^*),

$$P\left(T_{n_{m_k}} - T_{n_{m_{k-1}}} \geq 2 n_{m_k}^{\frac{1}{\alpha}} (\log n_{m_k})^{\frac{\varepsilon^*-\varepsilon}{\alpha}} \text{ i.o}\right) = 1$$

(15)

and

$$P\left(T_{n_{m_{k-1}}} \geq n_{m_k}^{\frac{1}{\alpha}}(\log n_{m_k})^{\frac{\varepsilon^*+\varepsilon}{\alpha}} \text{ i.o}\right) = 0 \qquad (16)$$

Define $z_n = n^{\frac{1}{\alpha}}(\log n)^{\frac{(\varepsilon^*-\varepsilon)}{\alpha}}$ and

$D_k = \left\{T_{n_{m_k}} - T_{n_{m_{k-1}}} \geq z_{n_{m_k}}\right\}, k \geq 1.$ Note that

$T_{n_{m_k}} - T_{n_{m_{k-1}}} \stackrel{d}{=} T_{n_{m_k} - n_{m_{k-1}}}, \forall\ k \geq 1.$ Hence by Theorem 1, one can find a constant $C_3>0$ and k_4 such that for all k ($\geq k_4$),

$$P(D_k) \geq C_3(n_{m_k} - n_{m_{k-1}})P\left(X \geq 2z_{n_{m_k}}\right)$$

$$\geq C_3 n_{m_k}\left(1 - \frac{n_{m_{k-1}}}{n_{m_k}}\right)P\left(X \geq 2z_{n_{m_k}}\right)$$

Since $\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$ implies that there exists λ<1 such that $\frac{n_{m_{k-1}}}{n_{m_k}} < \lambda < 1$, for all $k \geq k_4$,

$$P(D_k) \geq C_3 n_{m_k} P\left(X \geq 2z_{n_{m_k}}\right).$$

Now following the similar steps to those used to get an upper bound (13), one can find $C_4> 0$ and k_5 such that for all k ($\geq k_5$),

$P\left(D_k\right) \geq C_4(\log n_{m_k})^{\left(\varepsilon^* - \frac{\varepsilon}{2}\right)}.$ Note that $\sum_{k=k_4}^{\infty}(\log n_k)^{-\left(\varepsilon^* - \frac{\varepsilon}{2}\right)} = \infty$. Since D_k's are mutually independent and $\sum_{k=k_4}^{\infty} P(D_k) = \infty$ and by Borel-Cantelli lemma, we establish (15).

Again using Theorem 1, one can find a C_5 and a k_6 such that, for all $k \geq k_6$,

$$P\left(T_{n_{m_{k-1}}} \geq n_{m_k}^{\frac{1}{\alpha}}(\log n_{m_k})^{\frac{(\varepsilon^*-\varepsilon)}{\alpha}}\right)$$

$$\leq C_5 n_{m_{k-1}} P\left(X_1 \geq n_{m_k}^{\frac{1}{\alpha}}(\log n_{m_k})^{\frac{(\varepsilon^*-\varepsilon)}{\alpha}}\right)$$

Again following the steps similar to those used to get a lower bound of $P(M_k)$, one can find a constant $C_6>0$ and k_7 such that, for all k($\geq k_7$),

$$P\left(T_{n_{m_{k-1}}} \geq n_{m_k}^{\frac{1}{\alpha}}(\log n_{m_k})^{\frac{(\varepsilon^*-\varepsilon)}{\alpha}}\right)$$

$$\leq C_6 \frac{n_{m_{k-1}}}{n_{m_k}} \frac{1}{(\log n_{m_k})^{\left(\varepsilon^* - \frac{3\varepsilon}{2}\right)}}.$$

From (14), we infer that $n_{m_k} \geq \beta^{(k-1)^\delta}$ implies $n_{m_{k+1}} \geq \beta^{k^\delta} \geq n_{m_k}$ and since $\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$, there exists λ>1 such that $n_{k+1} \geq \lambda n_k$. Therefore,

$$n_{m_{k+1}} \geq \beta^{k^\delta} \geq n_{m_k} \geq \lambda n_{m_{k-1}} \Rightarrow$$

$$\lambda n_{m_{k-1}} \leq \beta^{k^\delta} \Rightarrow n_{m_{k-1}} \leq \lambda^{-1}\beta^{k^\delta} = \lambda_1\beta^{k^\delta},$$

where $\lambda_1 = \frac{1}{\lambda}$.

Hence

$\frac{n_{m_{k-1}}}{n_{m_k}} \leq \frac{\lambda_1\beta^{k^\delta}}{\beta^{(k-1)^\delta}} \cong \frac{\lambda_1}{\beta^{k^{\delta_1}}}$. We choose ε sufficiently small and by the definition of ε^*, we get

$$\sum_{k=k_7}^{\infty} \frac{n_{m_{k-1}}}{n_{m_k}} \frac{1}{(\log n_{m_k})^{\left(\varepsilon^* - \frac{3\varepsilon}{2}\right)}} \leq \lambda_1 \sum_{k=k_7}^{\infty} \frac{1}{\beta^{k^{\delta_1}}(\log n_{m_k})^{\left(\varepsilon^* - \frac{3\varepsilon}{2}\right)}} < \infty.$$

Therefore

$P\left(T_{n_{m_{k-1}}} \geq n_{m_k}^{\frac{1}{\alpha}}(\log n_{m_k})^{\frac{(\varepsilon^*-\varepsilon)}{\alpha}}\ i.o\right) = 0,$ which implies (13) follows from the proofs of (15) and (16) which completes the proof.

To prove (12), it suffices to show for any ε_2, $0 < \varepsilon_2 < 1$, that,

$$P\left(T_{n_k} \geq n_k^{\frac{1}{\alpha}} k^{\frac{(1+\varepsilon_2)}{\alpha}}\ i.o\right) = 0 \qquad (17)$$

and

$$P\left(T_{n_k} \geq n_k^{\frac{1}{\alpha}} k^{\frac{(1-\varepsilon_2)}{\alpha}}\ i.o\right) = 1 \qquad (18)$$

Observe that as the case

$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty$, as $k\to\infty$ comes under the class of at least geometrically increasingsubsequences, the proofs of

(17) and (18) follows on the similar lines of proofs of (15) and (16) and hence the details are omitted.

Theorem 3

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d positive r.v.s with a d.f F and assume that F ϵ DA(α), 0 < α < 1. Let $T_{n_k} = \sum_{k=1}^{n} f\left(\frac{k}{n_k}\right) X_{n_k}$, where f is a positive, non-decreasing and continuous function on [0,1]. Let $\{n_k\}$ be an integer subsequence such that $\operatorname{Lim\,sup}_{k\to\infty} \frac{n_{k+1}}{n_k} < \infty$.

Then

$$\operatorname{Lim\,sup}_{n\to\infty} \left(\frac{T_{n_k}}{n_k^{\frac{1}{\alpha}}}\right)^{\frac{1}{\log \log n_k}} = e^{\frac{1}{\alpha}} \text{ a.s.}$$

Proof

Proceeding as in Theorem, it is enough if we show that for any $\varepsilon_1 \epsilon (0,1)$,

$$P\left(T_{n_k} \geq n_k^{\frac{1}{\alpha}} (\log n_k)^{\frac{1+\varepsilon_1}{\alpha}} \text{ i.o} \right) = 0 \qquad (19)$$

and

$$P\left(T_{n_k} \geq n_k^{\frac{1}{\alpha}} (\log n_k)^{\frac{1-\varepsilon_1}{\alpha}} \text{ i.o} \right) = 1 \qquad (20)$$

One can note that (19) is a consequence of the theorem of Vasudeva[1978], that is

$$\operatorname{Lim\,sup}_{k\to\infty} \left(\frac{T_{n_k}}{n_k^{\frac{1}{\alpha}}}\right)^{\frac{1}{\log\log n_k}} \leq \operatorname{Lim\,sup}_{n\to\infty} \left(\frac{T_n}{B_n}\right)^{\frac{1}{\log\log n}} = e^{\frac{1}{\alpha}} \text{a.s.}$$

where B_n is a sequence of constants with $B_n > 0$ and $B_n = \inf\left\{x > 0 : 1-F(x)+F(-x) \geq \frac{1}{n}\right\}$. Since $\operatorname{Lim\,sup}_{k\to\infty} \frac{n_{k+1}}{n_k} < \infty$ we see that the sequences are at most geometrically increasing, which implies that there exists ϴ>1 such that $n_{k+1} \leq \theta n_k$. Now define where M is chosen such that $\frac{\theta}{M} < 1$. Proceeding as in Allan Gut [1986], one can show that $M^j < n_{v_j} < \theta M^j$ and $\frac{1}{\theta M} \leq \frac{n_{v_{j-1}}}{n_{v_j}} \leq \frac{\theta}{M} < 1$. Consequently, (n_{v_j}) satisfies the condition $\operatorname{Lim\,sup}_{j\to\infty} \frac{n_{v_{j-1}}}{n_{v_j}} < 1$ of Theorem and also the relation $\sum_{j=1}^{\infty} \left(\log n_{v_j}\right)^{-\varepsilon_1} < \infty$ holds for all $\varepsilon_1 > 1$ (i.e., $\varepsilon^* = 1$). Now (20) follows from the Theorem. Hence the proof of the theorem is completed.

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